16

FRACTIONAL KINETIC EQUATION (FKE)

Any kind of kinetic equation is an approximate way to describe an ensemble of trajectories or particles, while neglecting some details of dynamics. All this means that, depending upon the information about the system we would like to preserve, the type and specific structure of the kinetic equation depends on our choice of the reduced space of variables and on the level of coarse-graining of trajectories.

The origin of the *fractional kinetic equation* (FKE), or simply *fractional kinetics* (FK) is two-fold. First, it is based on the existence of singular zones in phase space that create a set of sticky domains. We can map the dynamics considering only parts of trajectories in sticky domains and neglecting the parts of trajectories of their transition from one sticky domain to another one. In such a way we define a new support of the reduced part of phase space. Second, the new support based on a set of sticky domains is of a fractal or multifractal structure, generally speaking, in time and in phase space (or configuration space) simultaneously. These properties of dynamics require a new approach to kinetics when the scaling features of the dynamics dominate others and, moreover, do not have a universal pattern as in the case of Gaussian processes, but instead, are specified by the phase space topology and the corresponding characteristics of singular zones.

This chapter consists of a specific approach to the kinetic description of Hamiltonian chaotic dynamics. The derived equation is called the Fractional Fokker–Planck–Kolmogorov (FFPK) equation, or simply FKE (*Note 16.1*).

Some elements of *fractional calculus* proved to be useful for FFPK and we put the necessary definitions and useful formulas in Appendices C and D (*Note 16.2*).

16.1 Derivation of FKE

A derivation of the FFPK consists of two steps: a formal phenomenological derivation of the equation and an establishment of the relations between the critical exponents. At the first step we follow a similar scheme as introduced by Kolmogorov for the FPK equation, with some modification (see Section 14.2). We use the same notation as in Section 14.2:

$$W(x, x_0; t, t_0) = W(x, x_0; t - t_0) = W(x, x_0; t) \equiv P(x, t) \qquad (16.1)$$

for the transition probability (14.11) with time uniformity (14.12), and the notation (14.15) for a convenience.

Let us start from a general expression for kinetic equations and let $\hat{\Delta}_t P(x,t)$ be a generalized infinitesimal shift of $P(x,t)$ along t by Δt. In a 'regular' case of the smooth variable t we have simply

$$\hat{\Delta}_t P(x,t) = P(x,t+\Delta t) - P(x,t) = \frac{\partial P(x,t)}{\partial t}\Delta t + O(\Delta t^2), \quad t > 0. \tag{16.2}$$

In the case of fractal time with fractal exponent β we have

$$\hat{\Delta}_t^\beta P(x,t) = \frac{\partial^\beta P(x,t)}{\partial t^\beta}(\Delta t)^\beta + O((\Delta t)^{\beta_1})), \qquad 0 < \beta \leq 1, \quad \beta_1 > \beta, \quad t > 0, \tag{16.3}$$

where the fractional derivative of order β has been introduced (see its definition in Appendix C). The explicit form of the shift difference operator $\hat{\Delta}_t^\beta$ is given in Appendix D. The main feature of (16.3) is that in the limit $\Delta t \to 0$ the right-hand side is proportional to $(\Delta t)^\beta$ and it shows a singular behaviour.

Let us now introduce an infinitesimal change of $P(x,t)$ due to transitions from other states $P(x',t)$ during the same time interval Δt. Since $\Delta t \to 0$, these transitions are local and they can be performed only from the points x' in the vicinity of x, assuming the absence of infinite velocities. A local structure of the phase space near a point x can be characterized by the fractal dimension α, and the corresponding changes $\hat{\Delta}_x^\alpha P(x,t)$ can be presented in a form similar to (14.16):

$$\hat{\Delta}_x^\alpha P(x,t) = \int dy\; W(x,y;\Delta t)P(y,t) - P(x,t) + O((\Delta t)^{\beta_2}), \qquad \beta_2 > \beta. \tag{16.4}$$

Since $P(x,t)dx$ can be interpreted as a number of particles at time t in the volume dx, the conservation of particles can be expressed as a balance equation

$$\hat{\Delta}_t^\beta P(x,t) = \hat{\Delta}_x^\alpha P(x,t) + O((\Delta t)^{\beta_3}), \qquad \beta_3 = \min(\beta_1, \beta_2) \tag{16.5}$$

or

$$\lim_{\Delta t \to 0} \frac{1}{(\Delta t)^\beta}\hat{\Delta}_t^\beta P(x,t) = \lim_{\Delta t \to 0} \frac{1}{(\Delta t)^\beta}\hat{\Delta}_x^\alpha P(x,t). \tag{16.6}$$

This expression is a formal representation of the FKE and its development depends on how the right-hand side will be calculated. Substituting (16.3)

and (16.4) into (16.6), we obtain

$$\frac{\partial^\beta P(x,t)}{\partial t^\beta} = \lim_{\Delta t \to 0} \frac{1}{(\Delta t)^\beta} \left\{ \int dy\, W(x,y;\Delta t) P(y,t) - P(x,t) \right\}, \qquad 0 < \beta \leq 1. \tag{16.7}$$

This equation is still exact and no assumption has been done about the properties of $W(x,y;\Delta t)$.

Let us assume now the existence of the expansion, similar to (14.19), in the limit $\Delta t \to 0$:

$$W(x,y;\Delta t) = \delta(x-y) + A(y;\Delta t)\delta^{(\alpha)}(x-y) + B(y;\Delta t)\delta^{(\alpha_1)}(x-y), \quad (0 < \alpha < \alpha_1 \leq 2) \tag{16.8}$$

with appropriate fractal dimension characteristics α and α_1. Beginning from (16.8), the approach to the kinetic equation becomes approximate. The approximation is in taking a finite number of terms in (16.8) and, what is more important, in assuming of the coefficients $A(y;\Delta t), B(y;\Delta t)$ independent on $P(x,t)$. Since the transition probability $W(x,y;\Delta t)$ represents the local features of the dynamics ($|x-y| \to 0$) and $P(x,t)$ represents strongly non-local features ($x, t \to \infty$), a physical nature of the assumption is independence of local transitions from the large time behaviour. At that point it is necessary to mention that the local-non-local independence hypothesis is the same for the FPK equation in Section 14.2 and here for the fractal space-time case. But the major difference is in the way the splitting of local-non-local distribution appears: for the FPK equation there exists a finite time t^* such that for $t > t^*$ one can assume an independence of $W(x,y;\Delta t)$ from the distribution $P(x,t)$, and there is no such t^* for fractional kinetic case as it will be clear later.

It is still possible to write the definition of the coefficient $B(y;\Delta t)$ through a moment of W:

$$\langle\langle |\Delta x|^{\alpha_1} \rangle\rangle \equiv \int dx |x-y|^{\alpha_1} W(x,y;\Delta t) = \Gamma(1+\alpha_1) B(y;\Delta t), \tag{16.9}$$

which is similar to (14.22), but the coefficient $A(y;t)$ does not have so simple an interpretation for the general case unless $B = 0$, or $\alpha_1 = \alpha + 1$.

By integrating (16.8) over y we obtain a relation

$$\frac{\partial^\alpha \mathcal{A}(x)}{\partial(-x)^\alpha} + \frac{\partial^{\alpha_1} \mathcal{B}(x)}{\partial(-x)^{\alpha_1}} = 0, \tag{16.10}$$

where

$$\mathcal{A}(x) = \lim_{\Delta t \to 0} \frac{A(x, \Delta t)}{(\Delta t)^\beta},$$
$$\mathcal{B}(x) = \lim_{\Delta t \to 0} \frac{B(y; \Delta t)}{(\Delta t)^\beta} = \frac{1}{\Gamma(1+\alpha_1)} \lim_{\Delta t \to 0} \frac{\langle\langle |\Delta x|^{\alpha_1} \rangle\rangle}{(\Delta t)^\beta} \tag{16.11}$$

similar to (14.25). Equation (16.10) is a generalization of (14.23) for the detailed balance principle. Existence of the limits (16.11) when $\Delta t \to 0$ is instead of the Kolmogorov conditions (16.25). Fractional values of α, α_1, and β represent a new type of the fractal properties of coarse-grained dynamics.

An important particular case is $\alpha_1 = \alpha + 1$, and (16.10) transforms into

$$\frac{\partial^\alpha}{\partial(-x)^\alpha} \left[\mathcal{A}(x) - \frac{\partial \mathcal{B}(x)}{\partial x} \right] = 0 \tag{16.12}$$

equivalent to (14.31), up to a constant in the notations, with

$$\mathcal{B}(x) = \frac{1}{\Gamma(2+\alpha)} \lim_{\Delta t \to 0} \frac{\langle\langle |\Delta x|^{\alpha+1} \rangle\rangle}{(\Delta t)^\beta},$$
$$\mathcal{A}(x) = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \frac{\langle\langle |\Delta x|^{\alpha} \rangle\rangle}{(\Delta t)^\beta}. \tag{16.13}$$

The generalization of the Landau formulas (14.23) and (14.24) is:

$$\Delta t \to 0: \quad \frac{\langle\langle |\Delta x|^{\alpha} \rangle\rangle}{(\Delta t)^\beta} = \frac{\Gamma(1+\alpha)}{\Gamma(2+\alpha)} \frac{\partial}{\partial x} \frac{\langle\langle |\Delta x|^{\alpha+1} \rangle\rangle}{(\Delta t)^\beta}. \tag{16.14}$$

The existence of the limits (16.13) can be considered as a generalized Kolmogorov condition (compare to (14.25)).

The FKE can be derived from (16.7) rewritten as:

$$\frac{\partial^\beta P(x,t)}{\partial t^\beta} = \lim_{\Delta t \to 0} \frac{1}{(\Delta t)^\beta} \left\{ \int dy [W(x, y; t + \Delta t) - \delta(x - y)] P(y, t) \right\}. \tag{16.15}$$

Using the expansion (16.8) and definitions (16.11), we obtain

$$\frac{\partial^\beta P(x,t)}{\partial t^\beta} = \frac{\partial^\alpha}{\partial(-x)^\alpha} (\mathcal{A}(x) P(x, y)) + \frac{\partial^{\alpha_1}}{\partial(-x)^{\alpha_1}} (\mathcal{B}(x) P(x, y)). \tag{16.16}$$

This equation will be called the Fractional Fokker–Planck–Kolmogorov equation (FFPK). It can be simplified in the case $\alpha_1 = \alpha + 1$

$$\frac{\partial^\beta P}{\partial t^\beta} = -\frac{\partial^\alpha}{\partial(-x)^\alpha} \left(\mathcal{B} \frac{\partial P}{\partial x} \right), \tag{16.17}$$

which transfers into regular diffusion equation for $\alpha = 1$ and $\mathcal{B} = \frac{1}{2}\mathcal{D}$.

Fractional derivatives are well defined in a specified direction (see the Appendix C), and there is no simple replacement $x \to -x$ or $t \to -t$. That is why a more general operator should be considered for the processes in $x \in (-\infty, \infty)$. For example, instead of the derivative of order α one can consider:

$$\hat{L}_x^{(\alpha)} = \frac{\mathcal{A}^+ \partial^\alpha}{\partial x^\alpha} + \frac{\mathcal{A}^- \partial^\alpha}{\partial (-x)^\alpha}. \tag{16.18}$$

For a symmetric case one can use Riesz derivative

$$\frac{\partial^\alpha}{\partial |x|^\alpha} = -\frac{1}{2\cos(\pi\alpha/2)} \left[\frac{\partial^\alpha}{\partial x^\alpha} + \frac{\partial^\alpha}{\partial (-x)^\alpha} \right], \qquad (\alpha \neq 1). \tag{16.19}$$

The corresponding FKE (16.16) takes the form (Saichev and Zaslavsky (1997))

$$\frac{\partial^\beta P}{\partial t^\beta} = \frac{\partial^\alpha}{\partial |x|^\alpha} (\mathcal{A}P) + \frac{\partial^{\alpha_1}}{\partial |x|^{\alpha_1}} (\mathcal{B}P), \qquad 0 < \alpha < \alpha_1 \leq 2. \tag{16.20}$$

In the case when the term with $\mathcal{B}$ can be neglected, we have a simplified version of FKE

$$\frac{\partial^\beta P}{\partial t^\beta} = \frac{\partial^\alpha}{\partial |x|^\alpha} (\mathcal{A}P). \tag{16.21}$$

In the case $\beta = 1$, $\alpha = 2$ it is a normal diffusion equation. For $0 < \beta < 1$, $\alpha = 2$

$$\frac{\partial^\beta P}{\partial t^\beta} = \frac{\partial^2}{\partial x^2} (\mathcal{A}P), \qquad (\beta < 1) \tag{16.22}$$

is called the equation of fractional Brownian motion (Mandelbrot and Van Ness (1968); Montroll and Shlesinger (1984)). For $\beta = 1$ and $1 < \alpha < 2$ the FKE corresponds to the Lévy process (see Section 15.2):

$$\frac{\partial P}{\partial t} = \frac{\partial^\alpha}{\partial |x|^\alpha} (\mathcal{A}P), \qquad (1 < \alpha < 2) \tag{16.23}$$

(*Note 16.3*).

Parameters β, α, α_1 will be called the *critical exponents*. They are subjected to be evaluated from the dynamics.

16.2 Conditions for the FKE

Any type of the FKE and its generalization can be considered as an independent mathematical problem. If we want to stay close to specific applications of the FKE to dynamical systems, restrictions related to the physical nature and the origin of the FKE should be imposed. Before considering solutions to the FKE, let us make a few comments about some constraints. Other conditions will be

presented later.

(a) *Interval condition.* We should define interval of consideration in space-time. Speaking about the space, we have in mind phase space (coordinate-momentum) and the variable x can represent any or both of them. The infinite intervals assume a possibility to have infinite moments of $P(x,t)$ while finite intervals $(x_{\min}, x_{\max})$, $(t_{\min}, t_{\max})$, that will be called *space* and *time windows*, lead to the finite moments since $P(x,t)$ is integrable.

(b) *Positiveness.* Solution $P(x,t)$ has the meaning of probability, and it should be positively defined, i.e.

$$P(x,t) \geq 0 \tag{16.24}$$

in the domain of consideration. For the infinite space-time domains condition (16.24) leads to restrictions on the possible values of critical exponents. Particularly we have the condition $0 < \alpha \leq 2$ for the Lévy processes ($\beta = 1$) or the condition:

$$0 < \beta \leq 1, \quad 0 < \alpha \leq 2 \tag{16.25}$$

(Saichev and Zaslavsky (1997)) for (16.21) with $\mathcal{A} = \text{const}$. A rigourous consideration of the $P(x,t)$ positiveness for both fractal values of (α, β) does not exist yet. Examples of the violation of (16.25) will be shown later.

(c) *Intermediate asymptotics.* It can be that dynamics imposes different asymptotics for different space-time windows. An example of two different asymptotics for particles advection in convective flow can be found in (Young *et al.* (1989)). Other examples will be indicated later. For such cases it should be different pairs (α_j, β_j) for different windows. Particularly, we should mention a multifractal situation when there are few different singular zones that impose different critical exponents for the FKE.

(d) *Definition of fractional integro-differentiation.* This definition is not unique (Samko *et al.* (1987)). We use the Riemann–Liouville form (see Appendix C) while sometimes other definitions may be more convenient. In fact, the structure of the FKE together with a type of the fractional derivative depends on the specific physical problem and the corresponding boundary and initial condition. More accurately, one can say that the distribution $P(x,t)$ is defined not only by the boundary-initial conditions but also by the type of derivatives used in FKE. A similar situation exists with Fourier or Laplace transforms. Any of them can be used if we know how to select a contour of integration in the complex plane to satisfy the boundary-initial conditions.

16.3 Evolution of moments (transport)

Let $P(x,t)$ be a solution for the FKE. The moments

$$\langle |x|^{\delta} \rangle = \int dx \; |x|^{\delta} P(x,t) \tag{16.26}$$

are the macroscopic observables. Their dependence on time defines the *transport*, i.e. the macroscopic evolution of the system.

It is easy to obtain the time-dependence of some moments if $\mathcal{A} = \text{const}$ in (16.21). Let us multiply the equation by $|x|^\alpha$ and integrate it over x. Then

$$\frac{\partial^\beta \langle |x|^\alpha \rangle}{\partial t^\beta} = \mathcal{A} \int dx \; |x|^\alpha \frac{\partial P(x,t)}{\partial |x|^\alpha}$$
$$= \mathcal{A} \int dx \; P(x,t) \frac{\partial^\alpha}{\partial |x|^\alpha} |x|^\alpha = \mathcal{A} \, \Gamma(1+\alpha), \tag{16.27}$$

where we use the formulas from the Appendix C. After integrating (16.27) over t^β (or differentiation with respect to $t^{-\beta}$) we obtain

$$\langle |x|^\alpha \rangle = \mathcal{A} \frac{\Gamma(1+\alpha)}{\Gamma(1+\beta)} t^\beta. \tag{16.28}$$

For the case of the self-similarity of the solution for FKE, which we will discuss more in Section 18.1, one may expect

$$\langle |x| \rangle \sim t^{\beta/\alpha} = t^{\mu/2}, \tag{16.29}$$

where we introduce the *transport exponent*

$$\mu \equiv \frac{2\beta}{\alpha}. \tag{16.30}$$

The meaning of the transport exponent is very simple: in the case that the second moment exists, i.e. $\langle x^2 \rangle < \infty$ and it is meaningful, we can write

$$\langle x^2 \rangle \sim t^\mu, \tag{16.31}$$

that is, for the normal diffusion it should be $\mu = 1$. The case

$$\mu = \frac{2\beta}{\alpha} > 1 \tag{16.32}$$

will be called *superdiffusion*, and the case

$$\mu = \frac{2\beta}{\alpha} < 1 \tag{16.33}$$

will be called *subdiffusion*.

It will be shown in the next chapter that the moments with $\delta > \alpha$ diverge. Nevertheless, there are some constraints on the application of the FKE to real dynamics, which we discuss in the following section.

16.4 Conflict with dynamics

Limitations for application of the FKE to dynamical systems can be compared to the limitations for diffusional process described in Section 14.6. Consider analogues (16.13) and (16.14) to the Kolmogorov condition (14.25):

$$\frac{(\delta x)^\alpha}{(\delta t)^\beta} = v^\alpha (\delta t)^{\alpha-\beta} = \mathcal{A} = \text{const}, \qquad (\delta t \to 0). \tag{16.34}$$

At the same time,

$$\alpha - \beta = \alpha\left(1 - \frac{\beta}{\alpha}\right) = \alpha\left(1 - \frac{\mu}{2}\right) > 0 \tag{16.35}$$

since $\mu < 2$. This means that in the limit $\delta t \to 0$ should be $v \to \infty$, and we arrive at the same conflict as in the normal diffusion case, that is, to the existence of infinite velocity in dynamics, which has no physical sense. A resolution of this conflict is similar to the case of normal diffusion: there exists $\delta t_{\min}$ such that for $\delta t < \delta t_{\min}$ the FKE cannot be applied.

More serious constraints are imposed by the condition of positiveness of $P_{\alpha,\beta}(x,t)$, i.e. $\beta < 1$, $0 < \alpha < 2$. Some simulations show, as we will see later, the values of $\alpha > 2$. A theory of the FKE is not developed yet for $\beta > 1$ and $\alpha > 2$.

We also need to consider truncated distribution function

$$P_{\alpha,\beta}(x,t) = \begin{cases} P^{(\mathrm{tr})}_{\alpha,\beta}(x,t), & 0 < x \le x_{\max}, \\ 0, & x > x_{\max} \end{cases} \tag{16.36}$$

(compare to (14.66)) and truncated moments

$$\langle |x|^m \rangle_{\mathrm{tr}} = \int dx \; P^{(\mathrm{tr})}_{\alpha,\beta}(x,t) < \infty \tag{16.37}$$

in order to avoid infinite velocities in the solutions, forbidden by the dynamics. All these comments will be necessary when real experimental or simulation data are compared to the theory. As it follows in the case of self-similarity

$$\langle |x|^m \rangle_{\mathrm{tr}} \sim t^{m\mu/2} \tag{16.38}$$

and all truncated moments are finite, although there is a restriction on the value of m which depends on $t_{\max}$ (see the discussion in Section 14.7). Particularly

$$\langle |x|^2 \rangle_{\mathrm{tr}} \sim t^{\mu} \tag{16.39}$$

introducing the transport exponent μ instead of (16.29).

For $\beta = 1$ we have the Lévy process with $\mu = 2/\alpha < 2$ since the second and higher moments ($\mu \ge 2$) diverge. The Lévy condition $\alpha < 2$ means that

$$1 < \mu < 2 \tag{16.40}$$

or that the permitted values of α are

$$1 < \alpha < 2 \tag{16.41}$$

and it is not clear from the dynamics why the values of $0 < \alpha < 1$ are not achievable. For the interval (16.40) $\mu > 1$, that is, the transport is superdiffusive, and it is not clear from the dynamics if subdiffusive transport with $\mu < 1$ is forbidden or not. The value $\mu = 2$ corresponds to the pure ballistic case. This will be discussed more later (*Note 16.4*).

All simulation data in the book will only be truncated moments and truncated distribution functions, although it will not be indicated explicitly.

16.5 Dynamical origin of critical exponents

In this section we demonstrate two examples of how the critical exponents can be obtained from the dynamical consideration of a model. More examples and speculations will be proposed in Chapter 17. These two examples are related to the web map and standard map (*Note 16.5*).

First, let us demonstrate the presence of the superdiffusion for the two maps. Figure 16.1 shows the behaviour of the second moment:

$$\mathcal{D} = \lim_{n\to\infty} \frac{1}{n}\langle R_n^2 \rangle = \lim_{n\to\infty} \frac{1}{n}\langle u_n^2 + v_n^2 \rangle$$

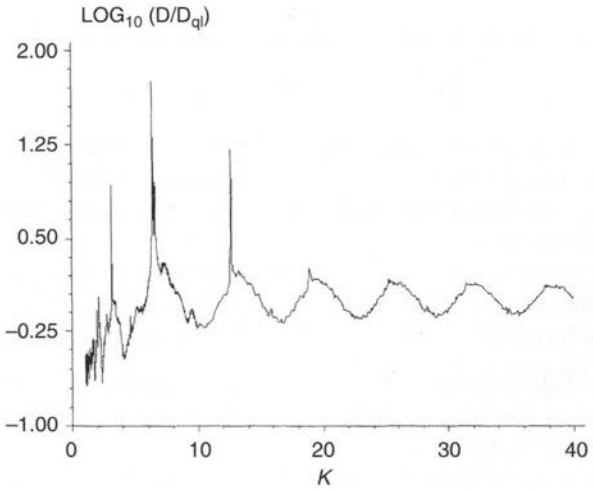

FIG. 16.1. Diffusion coefficient $\mathcal{D}$ from the simulation of the web map vs. K, normalized over the value $\mathcal{D}_{q\ell} = K^2/2$ that corresponds to the normal diffusion (so-called quasi-linear approximation).

for the web map. In the case of a normal diffusion $\mathcal{D} = \text{const} = \mathcal{D}_{q\ell} = K^2/2$. In the superdiffusive case

$$\langle R^2 \rangle = \langle u^2 + v^2 \rangle \sim t^{\mu} \tag{16.42}$$

with $\mu > 1$, that is, $\langle R^2 \rangle / t \to \infty$ as $t \to \infty$. Sharp peaks in Fig. 16.1 indicate the phenomenon of superdiffusion near special values of K. The larger t is, the sharper and higher are the peaks.

A set $\{K^{(j)}\}$ of the special values of K corresponds to the occurrence of accelerator mode islands of different resonance order (see Section 9.2), and there are intervals ΔK where the set of $K^{(j)} \in \Delta K$ is as dense as rationals. This property means that the topology of phase space is sensitive to the changes of control parameter K and, particularly, we arrive to the important conclusion of the dependence

$$\mu = \mu(K), \tag{16.43}$$

which is non-analytical (*Note 16.6*).

A similar example exists for the standard map (Fig. 16.2) for which the regular diffusion is governed by the equation

$$\frac{\partial F(p,t)}{\partial t} = \frac{1}{2}\mathcal{D}\frac{\partial^2 F(p,t)}{\partial p^2} \tag{16.44}$$

with $\mathcal{D} = \mathcal{D}_{q\ell} = K^2/2$. It follows from (16.44) that

$$\langle p^2 \rangle = \mathcal{D}t, \tag{16.45}$$

that is, $\langle p^2 \rangle / \mathcal{D}t = \text{const}$. In fact, simulations show the superdiffusion

$$\langle p^2 \rangle = \text{const} \cdot t^{\mu_p}, \qquad \mu_p > 1 \tag{16.46}$$

with $\mu_p = \mu_p(K)$ for some values of K. Figure 16.2 displays the values $\langle p^2 \rangle / t$ that have sharp peaks due to $\mu_p > 1$. For the normal diffusion $2\mathcal{D}/K^2 = 1$. The larger the time, the larger the peaks. They also appear due to the accelerator mode islands (*Note 16.7*):

These two examples show that there should be many different pairs (α, β) of critical exponents depending on the value of K. The way to observe the fractional kinetics can be formulated in the following scheme of assumptions:

(i) The value of the control parameter, say K, would be selected to have $t \to \infty$ domination of the only pair (α, β) and the corresponding fractal structure of phase space and flights.

(ii) Having only one scaling for time, say λ_T, and for space or phase space, say λ_ℓ, we link

$$\beta = \frac{1}{\ln \lambda_T}, \quad \alpha = \frac{1}{\ln \lambda_\ell}, \tag{16.47}$$

that is,

$$\mu = \frac{2\beta}{\alpha} = \frac{2 \ln \lambda_\ell}{\ln \lambda_T}. \tag{16.48}$$

(ii) The scaling constants λ_T, λ_ℓ can be obtained from any characteristics of space-time dynamics. For example, λ_T can be obtained from the scaling property of the Poincaré recurrences, or from the scaling of periods of islands-around-islands, and λ_ℓ can be obtained from the scaling parameter λ_S of the areas of islands-around-islands.

Let us show an example for the web map (Zaslavsky and Niyazov (1997)). Table 16.1 shows parameters of islands for a special value of $K = 6.34972$ that corresponds to the islands' hierearchy and the corresponding HIT $1-8-8-8-\cdots$ (see Section 12.2 and Fig. 12.4). The table is similar to Table 15.1:

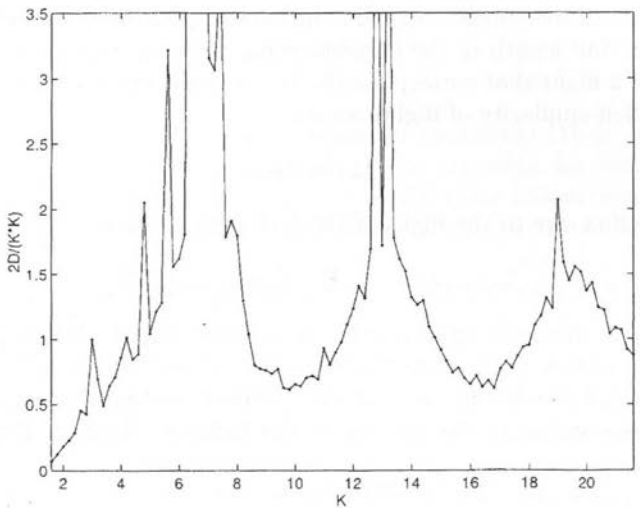

FIG. 16.2. Second moment $\langle p^2 \rangle / t$ dependence on K for the standard map.

TABLE 16.1 Parameters of self-similarity for the web map with $K = 6.349972$.

k	q_k	T_k	T_k/T_{k-1}	ΔS_k	$\Delta S_k/\Delta S_{k-1}$	δS_k	$\delta S_k/\delta S_{k-1}$
0	1	16.4	...	0.436	...	0.436	...
1	8	131.8	**8.04**	5.24×10^{-3}	$\mathbf{1.20} \times 10^{-2}$	4.19×10^{-2}	**0.0961**
2	8	1049	**7.96**	5.30×10^{-5}	$\mathbf{1.01} \times 10^{-2}$	3.39×10^{-3}	**0.0809**
3	8	8420	**8.02**	5.32×10^{-7}	$\mathbf{1.00} \times 10^{-2}$	2.72×10^{-4}	**0.0802**

Here the notations are the same as in Chapter 13, that is, q_k is a constant of proliferation of islands

$$q_n = \lambda_q = \text{const.}, \quad n > 0 \tag{16.49}$$

and for the considered case $\lambda_q = 8$;

$$\delta S_n = q_n \Delta S_n \tag{16.50}$$

(compare to (15.47)) and

$$\begin{aligned} \lambda_S &= \frac{\delta S_k}{\delta S_{k-1}} \approx \text{const} < 1, \\ \lambda_T &= \frac{T_k}{T_{k-1}} \approx \text{const} > 1. \end{aligned} \tag{16.51}$$

The values of λ_T, λ_S are evident from Table 16.1.

When a trajectory sticks near the boundaries of islands that are of the k-th generation, it rotates almost regularly in narrow annuluses around the islands. Let ℓ_k be the full length of the corresponding piece of trajectory, that is, ℓ_k is the length of a flight that corresponds to the stickiness to the islands of the k-th generation. Self-similarity of flights means

$$\ell_{k+1} = \lambda_\ell \ell_k. \tag{16.52}$$

The particle flux due to the flight of the k-th generation is

$$\mathcal{N}_k = \text{const} \cdot \ell_k d_k = \text{const} \cdot \ell_k \delta S_k^{1/2}, \tag{16.53}$$

where d_k is a diameter of the k-th generation island. When the particles (trajectories) switch their flights from being around the islands of k-th generation to being around the islands of a nearest generation, say $k \pm 1$, there should be preservation of the flux due to the Liouville theorem, that is,

$$\mathcal{N}_k = \text{const} \cdot \ell_k \delta S_k^{1/2} \approx \text{const} \tag{16.54}$$

or

$$\ell_0 \cdot \delta S_0^{1/2} \lambda_\ell^k \lambda_S^{k/2} \approx \text{const}. \tag{16.55}$$

It follows that

$$\lambda_\ell \approx \frac{1}{\lambda_S^{1/2}}. \tag{16.56}$$

Substitution of (16.56) to (16.48) gives the transport exponent as

$$\mu = \frac{|\ln \lambda_S|}{\ln \lambda_T} \tag{16.57}$$

and the corresponding FFPK equation

$$\frac{\partial^\beta P(\ell,t)}{\partial t^\beta} = \mathcal{A}\frac{\partial^\alpha P(\ell,t)}{\partial \ell^\alpha} \tag{16.58}$$

for the distribution function $P(\ell,t)$ of the flight length ℓ and time. The corresponding transport equation reads

$$\langle \ell^\alpha \rangle = \text{const} \cdot t^\beta. \tag{16.59}$$

The formula (16.57) shows how the dynamical features of trajectories, i.e. the scaling constant λ_S, λ_T, define the transport features—the fractal space-time exponents (α, β) and the transport exponent μ. All these conclusions can be compared to experiments or to simulations that will be discussed in the following section.

The final formulas (16.57) or (16.48) can be compared to (13.45) and (13.60) for the Poincaré recurrences exponent. We have an important connection for the case when the phase space topology provides a hierarchical set of sticky islands:

$$\gamma_{\text{rec}} = 2 + \mu = 2 + \frac{|\ln \lambda_S|}{\ln \lambda_T} = 2\left(1 + \frac{\ln \lambda_\ell}{\ln \lambda_T}\right). \tag{16.60}$$

Performing collection of data we can obtain the exponents (16.47) for the flight length and their time scaling and calculate the exponent for recurrences γ_{rec} or the transport exponent μ, and vice versa. All these characteristics are now linked. Particularly for the web map with the data given in Table 16.1, the phase space with a clear stickiness topology is in Fig. 12.4, a sample of trajectory is in Fig. 15.1, right, and the value of μ from Table 16.1 is $\mu \approx 1.21$, while the directly obtained value $\mu \approx 1.26$ with a good agreement (*Note 16.8*).

16.6 Principles of simulations

Comparison of any kind of theoretical prediction to the experimental or simulation data for fractal objects is very non-trivial, and here we discuss some major principles of the simulation performance. The reason for that is a specific kind of the randomness of trajectories when the Lyapunov exponent is small in singular zones of the phase space.

(a) *Non-universality.* It is always desirable in physics to have some universal constants for the most important phenomena. The only universal behaviour is the normal, or Gaussian, transport. Anomalous transport is not universal and the same is true of critical exponents α, β, and μ. Nevertheless, there are classes of universality which will be discussed later. It is always necessary to select a value of the control parameter K that defines a type of stickiness, dynamical trap, or singular zone.

(b) *Representativity of the ensemble of initial data.* The phase space and mixing is non-uniform in the case of stickiness of trajectories and presence of singular zones. To have a correct averaging $\langle \cdots \rangle$ over trajectories, their initial

conditions should be taken uniformly at the most uniform part of phase space to avoid influence of a fine structure of singular zones.

(c) *Non-ergodicity.* A specific value of the control parameter K^* defines a type of the singular zone in phase space. We assume that K^* is selected in such a way that the only singular zone dominates the transport. Nevertheless, the value of K^* can be selected only approximately and, due to that, for $t > t^*$ some other singular zones can influence the transport. Since the time of simulation is bounded from above and there is no finite time of relaxation to the uniform mixing, a long time of simulation does not provide a good approximation to the transport exponent, related to K^*, i.e. the value $\mu = \mu(K^*)$. The selection of large number of initial conditions rather than long time of observation can improve the situation.

(d) *Values of critical exponents.* It seems that a pure fractal situation with accurate values of critical exponents is a fairly rare phenomenon for at least two reasons: multiplicity of the islands' topology of phase space and log-periodicity discussed already in Section 10.6. A correct way is to introduce a spectral function for the critical exponents similar to the spectral function of multifractals (see in Zaslavsky (2000b)). Nevertheless, by a convenient choice of K^* one can create a situation close enough to the monofractality.

Notes

Note 16.1

The idea of exploiting fractional calculus and presenting kinetics in a form of an equation with fractional integro-differentiation is not new. For example, some variants of FK were used in Mandelbrot and Van Ness (1968) for signals; Young *et al.* (1989) for kinetics of advected particles; Hanson *et al.* (1985) for kinetics through cantori for the standard map; Nigmatullin (1986) for the porous media; Douglas *et al.* (1986, 1987) in macromolecules (see also a review paper by Douglas (2000)). For more recent publications, see Isichenko (1992) and Milovanov (2001) for the problem of percolation; Hilfer (1993, 1995a,b) for evolution and thermodynamics; West and Grigolini (2000) for time series. The material of this chapter is based on the results in Zaslavsky (1992, 1994a, 1994b), where FKE was derived with fractional derivatives in space and in time for the dynamics with singularities.

Note 16.2

For details on fractional calculus, see Gelfand and Shilov (1964); Samko *et al.* (1987); Miller and Ross (1993); Podlubny (1999); West *et al.* (2003). Different review articles and applications are collected in Hilfer (2000).

Note 16.3

There are different forms for the FKE that generalize forms (16.20)–(16.23): anisotropic equations were considered in Yanovsky *et al.* (2000); Meerschaert *et al.* (2001); equations with directional fractality in Weitzner

and Zaslavsky (2001); Meerschaert *et al.* (2001); non-linear fractional equations (Biler *et al.* (1998); Barkai (2001); Schertzer *et al.* (2001)). Some of these equations will be considered later.

We should mention that the investigation of the type and properties of the FKE is at its beginning stage and a number of important questions are not answered yet. Some of these questions will be discussed in the following sections.

Note 16.4

The conclusion on the absence of subdiffusion in Hamiltonian dynamics occurs after some assumption of applicability of the Lévy type process and the Kac lemma. Subdiffusion can appear along some axis while the whole random walk corresponds to the superdiffusion. We cannot exclude a possibility of the 'absolute' subdiffusion (in all directions) due to violation of some specific conditions of the lemma, for example, the absence of non-singular distributions.

Note 16.5

For the details of simulations, see Zaslavsky *et al.* (1997); Zaslavsky and Niyazov (1997); and Benkadda *et al.* (1997). More data and speculations around them will be shown later.

Note 16.6

This conjecture was proposed in Chernikov *et al.* (1990) after considering advection in a hexagonal variant of the so-called ABC-flow (see Section 23.1). The dependence of the type $\mu = \mu(K)$ was presented from simulation, and its singular dependence on K was linked to bifurcations and changes of the phase space topology.

Note 16.7

Peaks in diffusion for the standard map were noted in Ichikawa *et al.* (1987). Similar results with more details were also reported in Benkadda *et al.* (1997) and Zaslavsky *et al.* (1997).

Note 16.8

The data are given from Zaslavsky and Niyasov (1997). The directly obtained value μ means calculation of moments $\langle R^{2m} \rangle$ with $m = 1/2, 1, 2, \ldots$. We discuss them in Chapter 17. There are other simulations for different values of control parameters and different problems (Benkadda *et al.* (1997); Zaslavsky *et al.* (1997); White *et al.* (1998); Carreras *et al.* (1999); Carreras *et al.* (2001); Kuznetsov and Zaslavsky (1998, 2000); Leoncini *et al.* (2001); and others).

Problems

More complicated problems are marked by (*).

16.1 Starting from (16.58), calculate the constant in (16.59).